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ON 3-DIMENSIONAL CONTACT METRIC MANIFOLDS

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Abstract

Let M be a 3-dimensional almost contact metric manifold satisfying ($*$)-condition. We denote such a manifold by M^*. We prove that if M^* is η-Einstein, then M^* is either Sasakian or cosymplectic manifold, and is a space of constant curvature. Consequently M^* is either flat or isometric to the 3-dimensional unit sphere if M^* is complete and simply connected.

1. Introduction

The conformal curvature tensor C is invariant under conformal transformations and vanishes identically for 3-dimensional manifolds. Using this fact many authors [1, 3, 4, 6] studied 3-dimensional almost contact manifolds. In [5], they introduced a new class of almost contact manifold M^* containing quasi-Sasakian and trans-Sasakian structure. Moreover they constructed non-trivial examples. In this paper, we study a 3-dimensional η-Einstein manifold M^* by use of the fact that C vanishes identically and the special form of Ricci curvature. Consequently, we prove that the 3-dimensional η-Einstein manifold M^* becomes either Sasakian or cosymplectic manifold, and is a space of constant curvature. In the cosymplectic case, M^* is flat, and if M^* is Sasakian, complete and simply connected, then M^* is isometric to the 3-dimensional unit sphere, that is M^* is either flat or isometric to $S^3(1)$ under this topological condition.

2. Almost contact metric structure

Let M be an m-dimensional real differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$, in which there are given a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η satisfying

$$(2.1) \qquad \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

for any vector field X on M. Such a set of (ϕ, ξ, η) is called an *almost contact structure* and we call a manifold with an almost contact structure an *almost contact manifold*. In an almost contact manifold, if there is given a Riemannian metric g such that

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$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y on M, we say M has an *almost contact metric structure* and g is called a compatible metric. Setting $Y = \xi$, we have immediately $\eta(X) = g(X, \xi)$.

The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(\phi X, Y)$. It is known that the almost contact structure (ϕ, ξ, η) is normal if and only if the Nijenhuis tensor

$$N(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi$$

vanishes, where [,] is a bracket operation and d denotes the exterior derivative. An almost contact metric structure (ϕ, ξ, η, g) on M is said to be

(a) *Sasakian* if $\Phi = d\eta$ and (ϕ, ξ, η) is normal,

(b) *cosymplectic* if Φ and η are closed and (ϕ, ξ, η) is normal.

In [5], one of the present author defined a new class of almost contact metric structure on M which satisfies

$$(*) \qquad d\Phi = 0, \quad \nabla_X \xi = \lambda \phi X \text{ and } (\phi, \xi, \eta) \text{ is normal}$$

for a smooth function λ on M and ∇ denotes the Riemannian connection for g. Briefly, we denote such a manifold by M^*. It is easily seen that M^* is cosymplectic if $\lambda = 0$, and Sasakian if λ is a non-zero constant.

Theorem 1 [5]. *On M^*, we have*

$$(\nabla_X \phi)(Y, Z) = \lambda\{\eta(Y)g(X, Z) - \eta(Z)g(X, Y)\}, \tag{2.2}$$

$$R(X, \xi)Y = (X\lambda)(\phi Y) + \lambda^2\{\eta(Y)X - g(X, Y)\xi\}, \tag{2.3}$$

$$\xi\lambda = 0, \tag{2.4}$$

$$S(\xi, X) = (\phi X)\lambda + (m-1)\lambda^2\eta(X), \tag{2.5}$$

where S is the Ricci curvature tensor and R is the curvature tensor defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z.$$

3. 3-dimensional almost contact manifolds

Let M^* be a 3-dimensional manifold satisfying $(*)$. It is well known [2] that the conformal curvature tensor of Weyl vanishes identically for 3-dimensional manifolds. Therefore the curvature tensor R of a 3-dimensional manifold M^* is given by

$$\begin{aligned} R(X, Y)Z = &-S(X, Z)Y + S(Y, Z)X - g(X, Z)QY \\ &+ g(Y, Z)QX + \frac{r}{2}\{g(X, Z)Y - g(Y, Z)X\}, \end{aligned} \tag{3.1}$$

where r is the scalar curvature and Q is defined by $g(QX,Y)=S(X,Y)$. Using (2.3), (2.5) and (3.1), we have

$$S(X,Y)=\eta(X)(\phi Y)\lambda+\eta(Y)(\phi X)\lambda +\left(\frac{r}{2}-\lambda^2\right)g(X,Y)+\left(3\lambda^2-\frac{r}{2}\right)\eta(X)\eta(Y). \tag{3.2}$$

If we substitute (3.2) into (3.1), then we get

$$\begin{aligned}R(X,Y,Z,W)&=g(R(X,Y)Z,W)\\ &=-\eta(X)((\phi Z)\lambda)g(Y,W)-\eta(Z)((\phi X)\lambda)g(Y,W)\\ &\quad+\eta(Y)((\phi Z)\lambda)g(X,W)+\eta(Z)((\phi Y)\lambda)g(X,W)\\ &\quad-\eta(Y)((\phi W)\lambda)g(X,Z)-\eta(W)((\phi Y)\lambda)g(X,Z)\\ &\quad+\eta(X)((\phi W)\lambda)g(Y,Z)+\eta(W)((\phi X)\lambda)g(Y,Z)\\ &\quad+\left(2\lambda^2-\frac{r}{2}\right)\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\}\\ &\quad+\left(\frac{r}{2}-3\lambda^2\right)\{(\eta(X)g(Y,W)-\eta(Y)g(X,W))\eta(Z)\\ &\quad+(\eta(Y)g(X,Z)-\eta(X)g(Y,Z))\eta(W)\}.\end{aligned} \tag{3.3}$$

If we put $Y=\xi$ in (3.3), then by (2.3) we obtain

$$\begin{aligned}(X\lambda)\Phi(Z,W)&+\lambda^2\{\eta(Z)g(X,W)-\eta(W)g(X,Z)\}\\ &=\lambda^2\{\eta(Z)g(X,W)-\eta(W)g(X,Z)\}\\ &\quad+((\phi W)\lambda)\{\eta(X)\eta(Z)-g(X,Z)\}\\ &\quad-((\phi Z)\lambda)\{\eta(X)\eta(W)-g(X,W)\},\end{aligned} \tag{3.4}$$

that is

$$(X\lambda)\Phi(Z,W)=((\phi W)\lambda)\{\eta(X)\eta(Z)-g(X,Z)\} -((\phi Z)\lambda)\{\eta(X)\eta(W)-g(X,W)\}, \tag{3.5}$$

or in local components

$$\lambda_k\Phi_{ih}=\phi_h{}^t\lambda_t(\eta_i\eta_k-g_{ik})-\phi_i{}^t\lambda_t(\eta_h\eta_k-g_{hk}), \tag{3.6}$$

where $\lambda_k=\partial_k\lambda$ and the indices i, j, k, t run over the range $\{1,2,...,m\}$. From (3.5) or (3.6), we can calculate

$$\|\nabla_k\Phi_{ij}\|^2=(\lambda_k\Phi_{ij})(\lambda^k\Phi^{ij})=4\|\lambda_t\|^2-2\|\phi_i{}^t\lambda_t\|^2, \tag{3.7}$$

where $\lambda^k=g^{ik}\lambda_i$. Moreover we can easily see that

$$\|\phi_i{}^t\lambda_t\|^2=\|\lambda_t\|^2.$$

Lemma 2. *In a 3-dimensional manifold M^*, the function λ is constant if and only if $(\phi X)\lambda=0$ for all X.*

If the Ricci curvature S on M is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.8}$$

then M is called an η-Einstein space [1,6,7]. If M^* is η-Einstein, then we have

$$3a + b = r \tag{3.9}$$

and

$$a + b = r - 4\lambda^2 \tag{3.10}$$

by use of (2.1), (3.2) and (3.8). Hence we get $a = 2\lambda^2$ and $b = r - 6\lambda^2$. Therefore the Ricci curvature S becomes

$$S(X,Y) = (2\lambda^2)g(X,Y) + (r - 6\lambda^2)\eta(X)\eta(Y). \tag{3.11}$$

If we put $Y = \xi$ in (3.11), then we get

$$(\phi X)\lambda = (r - 6\lambda^2)\eta(X) \tag{3.12}$$

from (2.5) and (3.11). If we set $X = \xi$ in (3.12), then it gives

$$r = 6\lambda^2, \tag{3.13}$$

that is

$$(\phi X)\lambda = 0 \tag{3.14}$$

and that

$$S(X,Y) = 2\lambda^2 g(X,Y) \tag{3.15}$$

from (3.11). We see that λ is constant from Lemma 2 and (3.14). Since 3-dimensional Einstein space is a space of constant curvature, we obtain the following theorem by using Lemma 2, (3.14) and (3.15).

Theorem 3. *Let M^* be a 3-dimensional η-Einstein manifold. Then M^* is a space of constant curvature. Moreover M^* is either Sasakian or cosymplectic manifold.*

In case $\lambda = 0$, since M^* is a space of constant curvature, we have $r = 0$ and hence $R(X,Y)Z = 0$, that is M^* is flat.

On the other hand, E. M. Moskal obtained the following result (cf. [7]).

Theorem 4. *Let M be a complete and simply connected Sasakian manifold. If M is Einstein and of positive curvature, then it is isometric to the unit sphere.*

If λ is non-zero constant, then M^* is Sasakian. Therefore this fact and Theorems 3 and 4 reduce

Theorem 5. *Let M^* be a 3-dimensional η-Einstein manifold. Then M^* is either flat or isometric to $S^3(1)$ if M^* is complete and simply connected.*

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